# Theoretical Study of the Dick Effect in a Continuously Operated Ramsey Resonator

Alain Joyet, Gaetano Mileti, Gregor Dudle, and Pierre Thomann, *Member, IEEE*

*Abstract*—It is well established that passive frequency standards operated in pulsed mode may suffer a degradation of their frequency stability due to the frequency (FM) noise of the Local Oscillator (LO).

In continuously operated frequency standards, it has been shown that a similar degradation of the frequency stability may arise, depending on the used modulation-demodulation scheme. In this paper, we report a theoretical analysis on the possible degradations of the frequency stability of a continuous fountain due to the LO FM noise. A simple model is developed to evaluate whether or not aliasing persists. This model is based on a continuous frequency control loop of a frequency standard using a Ramsey resonator.

From this model, we derive a general formula, valid for all usual modulation-demodulation schemes, for the LO frequency fluctuations due to aliasing in closed loop operation. We demonstrate that in an ideal situation and for all usual modulation waveforms, no aliasing occurs if the half-period of modulation equals the transit time of atoms in the Ramsey resonator. We also deduce that in the same conditions, square-wave phase modulation provides the strongest cancellation of the LO instabilities in closed loop operation. Finally, we show that the "Dick formula" for the specific case of the pulsed fountain can be recovered from the model by a sampling operation.

*Index Terms*—Dick effect, frequency standards, laser cooling.

## I. INTRODUCTION

IN PASSIVE frequency standards operated in pulsed mode, like ion traps or cold atom fountains, it has been theoretically [1]–[3] and experimentally [4] demonstrated that this particular mode of operation may lead to a degradation of the frequency stability. In these kinds of devices, the frequency (FM) noise of the Local Oscillator (LO) around harmonics of the pulse rate is downconverted by aliasing into the bandpass of the frequency control loop. This mechanism, called "Dick effect" in the literature because it was first predicted and described by Dick at JPL [2], limits the achievable stability even with state-of-the-art quartz local oscillators.

In continuously operated frequency standards, a similar degradation of the frequency stability may arise. It depends on the scheme of modulation-demodulation used to generate the error signal which controls the LO frequency and on the value of the modulation frequency. This mechanism, the so-called

"intermodulation effect" was first pointed out by Kramer [1], then described in detail by Audoin [5] in the case of a passive cell standard in the quasi-static approximation (modulation frequency much smaller than atomic resonance linewidth).

We are interested here in possible degradations, due to the LO FM noise, of the frequency stability of a Ramsey resonator fed by a continuous beam of atoms showing a constant intensity. In this type of standards, atoms are continuously interrogated and any LO phase step will be detected and will produce an error signal. It is expected that this absence of dead times will constitute a significant advantage over the pulsed fountains, where aliasing is unavoidable.

A generalization of the Dick formalism for pulsed Ramsey resonators to the continuous case was proposed in [6], [7], and briefly addressed in [8]. Our approach is complementary to [7], [8] in that we start out with the error signal generated in the continuous case and derive the "Dick formula" as a particular case where the continuous error signal is sampled.

To evaluate whether aliasing nevertheless persists, we consider a simple model, described in Section II, of a continuously operated frequency standard using a Ramsey resonator. In Section III, the error signal of the frequency control loop is derived in a general case involving any modulation-demodulation waveform. Then, we calculate in Section IV the frequency spectrum of the Locked Local Oscillator (LLO) when the loop is closed from the knowledge of the power spectral density $S_y^{\mathrm{LO}}(f)$ of fractional frequency fluctuations of the unlocked LO. Finally, we deduce the LLO power spectral density $S_{y,\,al.}^{\mathrm{LLO}}(f)$ of fractional frequency fluctuations due to aliasing. The result constitutes a formula valid for continuous fountain Ramsey resonators and similar to the "Dick formula" which was derived specifically for pulsed resonators. Then, we discuss its main features in Section V as well as the limitations of the model. Finally, we show in the last section that it is possible to recover the "Dick formula" for the specific case of the pulsed fountain by a sampling operation.

## II. MODEL OF FREQUENCY CONTROL LOOP

In the present derivation of the LLO frequency stability limitation, we focus our attention on passive fountain frequency standards using a continuous Ramsey interrogation scheme [9], [10]. A sketch of the considered continuous fountain frequency standard is represented in Fig. 1.

We assume a conventional quartz crystal oscillator characterized by its power spectral density of fractional FM noise $S_y^{\mathrm{LO}}(f)$. The atomic resonator is supposed to be an ideal Ramsey resonator with two infinitely short Rabi pulses. The
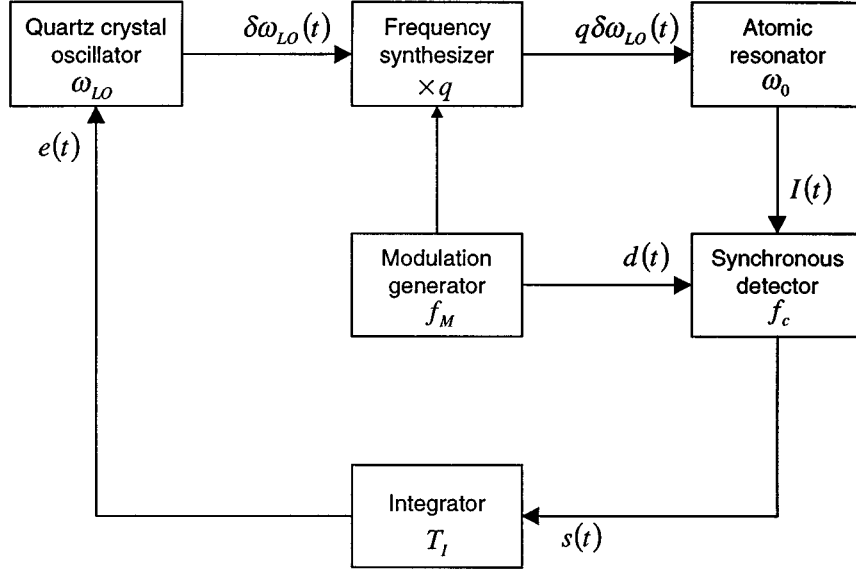
Fig. 1.   Block-diagram of the continuous fountain frequency standard.

signal at the output of the resonator is synchronously detected at a modulation frequency $f_M$. Then, after integration, it generates the continuous error signal $e(t)$ used to control the LO frequency. In order to evaluate the specific contribution of the Dick effect to the frequency instability, we do not take into account the shot noise in the atomic beam and we also assume that the Ramsey resonator, the frequency synthesizer, the synchronous detector, and the modulation generator add no noise to the error signal.

### III. ERROR SIGNAL OF THE FREQUENCY CONTROL LOOP

We suppose that the Ramsey resonator is fed by a signal of the form

$$V(t) = V_0 \cos(\omega t + \phi(t)). \tag{1}$$

The atoms are then subjected to two oscillatory magnetic fields of constant amplitude and of angular frequency $\omega$, the first with phase $\phi(t - T)$ and the second with phase $\phi(t)$. $T$ is the transit time between the two Rabi pulses and $\omega$ is the mean angular frequency of the interrogation signal which is a multiple of the LO angular frequency $\omega_{LO}$. The time-dependent phase $\phi(t)$ contains several terms

$$\phi(t) = \phi_{\text{mod}}(t) + q\delta\phi_{LO}(t) + \phi_0. \tag{2}$$

The first one, $\phi_{\text{mod}}(t)$, is a general phase modulation waveform needed to generate the error signal which will allow the frequency control loop to keep the angular frequency $\omega$ as close as possible to that of the cesium atom. $q$ is the multiplying factor of the frequency synthesizer and $\delta\phi_{LO}(t)$ represents the instantaneous LO phase fluctuations. $\phi_0$ is an arbitrary constant phase.

In the limit of infinitely short Rabi pulses, the phase of the interrogation microwave field is constant during the interaction. Therefore, the time-dependent probability that a transition occurs inside the Ramsey resonator between the hyperfine states

$F = 3$, $m_F = 0$ and $F = 4$, $m_F = 0$ of the cesium atom is given in a good approximation, valid for the central fringe of the Ramsey pattern, by [11]:

$$P(t) = \tfrac{1}{2} \left\{ 1 + \cos\left[ (\omega - \omega_0)T + \phi(t, T) \right] \right\} \tag{3}$$

for monokinetic atoms and optimum Rabi pulses ($\pi/2$). $\omega_0$ is the hyperfine transition angular frequency of the cesium atom and $\phi(t, T)$ is the instantaneous apparent phase difference experienced by the atoms between the two Rabi pulses, resulting from the effect of the phase modulation, the LO phase fluctuations and the motion of cesium atoms inside the resonator. It is given by

$$\phi(t, T) = \phi(t) - \phi(t - T). \tag{4}$$

Inserting (2) into (4) leads to the following expression:

$$\phi(t, T) = \phi_{\text{mod}}(t, T) + \delta\phi_{LO}(t, T) + \Delta\phi. \tag{5}$$

The first term is due to the generated phase modulation

$$\phi_{\text{mod}}(t, T) = \phi_{\text{mod}}(t) - \phi_{\text{mod}}(t - T) \tag{6}$$

where $\phi_{\text{mod}}(t - T)$ and $\phi_{\text{mod}}(t)$ are the instantaneous modulation phase in the first, respectively the second Rabi pulse. For the modulation waveforms generally used in frequency standards, $\phi_{\text{mod}}(t)$ is assumed to be an odd and periodic function of period $T_M$ whose spectrum contains only odd harmonics of the modulation frequency $f_M = 1/T_M$. As a result, $\phi_{\text{mod}}(t, T)$ can be written as

$$\phi_{\text{mod}}(t, T) = 2\phi_m c(t, T) \tag{7}$$

where $c(t, T)$ is a periodic function of period $T_M$ whose spectrum contains odd harmonics only and such that the maximum of $|c(t, T)|$ is unity. $\phi_m$ is the phase modulation depth of the modulation waveform $\phi_{\text{mod}}(t)$.

The second term in (5) is a random component due to the LO phase fluctuations

$$\delta\phi_{\mathrm{LO}}(t, T) = q\left(\delta\phi_{\mathrm{LO}}(t) - \delta\phi_{\mathrm{LO}}(t - T)\right) \tag{8}$$

where $\delta\phi_{\mathrm{LO}}(t-T)$ and $\delta\phi_{\mathrm{LO}}(t)$ are the instantaneous LO phase in the first, respectively the second Rabi pulse. This term can be directly expressed as a function of frequency fluctuations

$$\delta\phi_{\mathrm{LO}}(t, T) = q\int_{t-T}^{t} \delta\omega_{\mathrm{LO}}(t')\, dt' \tag{9}$$

where $\delta\omega_{\mathrm{LO}}(t)$ is a random function representing the LO angular frequency fluctuations.

The third term in (5) is a constant phase difference

$$\Delta\phi = \phi_2 - \phi_1 \tag{10}$$

where $\phi_1$ and $\phi_2$ are the arbitrary constant LO phase in the first, respectively, the second Rabi pulse.

In the following, we restrict our analysis to the case where the resonance condition is fulfilled, i.e., we assume that the frequency control loop keeps the mean detuning $\omega - \omega_0$ equal to zero. The remaining small time-dependent detuning is then only due to the LO frequency fluctuations around $\omega$. We also assume that there is no frequency independent background atom flux in the resonator. Therefore, the time-dependent response of the resonator to the phase modulation, which is proportional to the time-dependent Ramsey probability (3), is given by the signal of a photodetector at the output of the resonator and may be written as

$$I(t) = \tfrac{1}{2} I_0 \left\{1 + \cos\left[\phi(t, T)\right]\right\} \tag{11}$$

where $I_0$ is the signal at resonance if $\phi(t, T) = 0$. Inserting (5) and (7) into (11) and assuming that no permanent phase difference exists between the two Rabi pulses ($\Delta\phi = 0$), we have

$$\frac{I(t)}{I_0} = \frac{1}{2}\left\{1 + \cos\left[2\phi_m c(t, T) + \delta\phi_{\mathrm{LO}}(t, T)\right]\right\}. \tag{12}$$

The standard deviation of $\delta\phi_{\mathrm{LO}}(t, T)$ is much smaller than $2\phi_m$ which is usually of the order of $\pi/2$ radian

$$\delta\phi_{\mathrm{LO}}(t, T) \ll 2\phi_m. \tag{13}$$

We then can expand (12) limited to the first order of the ratio $\delta\phi_{\mathrm{LO}}(t, T)/2\phi_m$ to get

$$\frac{I(t)}{I_0} \cong \frac{1}{2}\left\{1 + \cos\left[2\phi_m c(t, T)\right]\right\}$$
$$- \frac{1}{2}\left(\frac{\delta\phi_{\mathrm{LO}}(t,T)}{2\phi_m}\right) 2\phi_m \sin\left[2\phi_m c(t, T)\right]. \tag{14}$$

This signal is then synchronously detected at $f_M$ and low-pass filtered with cutoff frequency $f_c \ll f_M$. Since the spectrum of the periodic function $c(t, T)$ contains only odd harmonics, it follows that the spectrum of $\cos[2\phi_m c(t, T)]$ and of $\sin[2\phi_m c(t, T)]$ contains only even and odd harmonics respectively. Therefore, only the third term of (14) will provide the error signal.

If $d(t)$ denotes the demodulation waveform of the synchronous detector, the signal at its output is given by

$$s(t) = -\tfrac{1}{2} K\delta\phi_{\mathrm{LO}}(t, T)\sin\left[2\phi_m c(t, T)\right] d(t) \tag{15}$$

where $K$ is a constant which depends on the synchronous detector gain. The error signal $e(t)$ applied to the LO is obtained after passing through the integrator

$$e(t) = \frac{1}{T_I}\int_{-\infty}^{t} s(t')\, dt' \tag{16}$$

where $T_I$ is the time constant of the integrator. Equations (15) and (16) are the basic relationships of our simple model and will be analyzed below in order to find the power spectral density of the LLO frequency fluctuations.

## IV. FREQUENCY SPECTRUM OF THE LLO

We are now interested to calculate the spectrum of the error signal defined by (15) and (16). The signal $s(t)$ at the output of the synchronous detector is composed of two parts:

- a random part $\delta\phi_{\mathrm{LO}}(t, T)$ depending on the LO angular frequency fluctuations $\delta\omega_{\mathrm{LO}}(t)$. We assume in the following that the process $\delta\omega_{\mathrm{LO}}(t)$ is stationary and of zero mean value;
- a deterministic part $\sin[2\phi_m c(t, T)]d(t)$ which depends on the modulation-demodulation scheme.

Let us start by calculating the two-sided power spectral density $S'_{\delta\phi_{\mathrm{LO}}}(f)$ of the random part $\delta\phi_{\mathrm{LO}}(t, T)$ given by

$$\delta\phi_{\mathrm{LO}}(t, T) = q\int_{t-T}^{t}\delta\omega_{\mathrm{LO}}(t')\, dt' = q\delta\omega_{\mathrm{LO}}(t) * h(t) \tag{17}$$

where the symbol $*$ stands for convolution and $h(t)$ is a rectangular impulse equal to 1 for $0 \le t \le T$ and 0 elsewhere. The autocorrelation function $R_{\delta\phi_{\mathrm{LO}}}(\tau)$ of $\delta\phi_{\mathrm{LO}}(t, T)$ reads then

$$R_{\delta\phi_{\mathrm{LO}}}(\tau) = q^2 R_{\delta\omega_{\mathrm{LO}}}(\tau) * h(-\tau) * h(\tau) \tag{18}$$

where $R_{\delta\omega_{\mathrm{LO}}}(\tau)$ is the autocorrelation function of $\delta\omega_{\mathrm{LO}}(t)$. The two-sided power spectral density is obtained by taking the Fourier transform of $R_{\delta\phi_{\mathrm{LO}}}(\tau)$. We get

$$S'_{\delta\phi_{\mathrm{LO}}}(f) = q^2\left|H(f)\right|^2 S'_{\delta\omega_{\mathrm{LO}}}(f) \tag{19}$$

where $H(f)$ is the transfer function of $h(t)$ and $S'_{\delta\omega_{\mathrm{LO}}}(f)$ is the two-sided power spectral density of the LO angular frequency fluctuations.

Let $a(t)$ be the deterministic part of the error signal:

$$a(t) = \sin\left[2\phi_m c(t, T)\right] d(t). \tag{20}$$

As the spectrum of the demodulation waveform $d(t)$ is assumed to contain only odd harmonics of the modulation frequency $f_M$, the spectrum of $a(t)$ will contain even harmonics only. Let us define the Fourier series development of $a(t)$

$$a(t) = c_0 + \sum_{k=1}^{+\infty} c_{2k} e^{i4k\pi f_M t} + c.c. \tag{21}$$

where the coefficients $c_{2k}$ are in principle calculable from the knowledge of the modulation and demodulation waveforms. Its autocorrelation function $R_a(\tau)$ is then the following:

$$R_a(\tau) = c_0^2 + \sum_{k=1}^{+\infty} |c_{2k}|^2 \left( e^{i4k\pi f_M\tau} + e^{-i4k\pi f_M\tau} \right). \quad (22)$$

The two-sided power spectral density follows immediately:

$$S_a'(f) = c_0^2\delta(f) + \sum_{k=1}^{+\infty} |c_{2k}|^2 \left[ \delta(f - 2kf_M) + \delta(f + 2kf_M) \right] \quad (23)$$

with $\delta(f)$ the Dirac impulse function.

The autocorrelation function of the signal $s(t)$ at the output of the synchronous detector is given by

$$R_s(\tau) = \frac{K^2}{4} R_{\delta\phi_{\mathrm{LO}}}(\tau)R_a(\tau) \quad (24)$$

since $\delta\phi_{\mathrm{LO}}(t, T)$ and $a(t)$ are two independent variables [12]. After filtering through an ideal lowpass filter (transfer function $H_F(f) = 1$ for $|f| < f_c$ and 0 elsewhere), the two-sided power spectral density $S_s'(f)$ of $s(t)$ is then readily obtained

$$S_s'(f) = \frac{K^2}{4} S_{\delta\phi_{\mathrm{LO}}}'(f) * S_a'(f) \qquad |f| < f_c. \quad (25)$$

At the output of the integrator and overlooking any mathematical complications, the two-sided power spectral density $S_e'(f)$ of the error signal $e(t)$ may be written as

$$S_e'(f) = \frac{S_s'(f)}{(2\pi f T_I)^2} \qquad |f| < f_c. \quad (26)$$

Inserting (19), (23), and (25) into (26), we obtain the following expression:

$$S_e'(f) = \frac{K^2 q^2}{16\pi^2 f^2 T_I^2} c_0^2 |H(f)|^2 S_{\delta\omega_{\mathrm{LO}}}'(f) + \frac{K^2 q^2}{16\pi^2 f^2 T_I^2} \sum_{k=1}^{+\infty}$$
$$\times \left\{ |c_{2k}|^2 \times \left[ |H(f - 2kf_M)|^2 S_{\delta\omega_{\mathrm{LO}}}'(f - 2kf_M) \right. \right.$$
$$\left. \left. + |H(f + 2kf_M)|^2 S_{\delta\omega_{\mathrm{LO}}}'(f + 2kf_M) \right] \right\}$$
$$|f| < f_c. \quad (27)$$

This spectrum is composed of two parts. The first one which contains $S_{\delta\omega_{\mathrm{LO}}}'(f)$ corresponds to the error signal that actually controls the LO frequency fluctuations. The second one, which contains all the even multiples of $f_M$, corresponds to a spurious error signal generated by downconversion of LO frequency fluctuations at higher harmonics of the modulation frequency, this is the aliased part of the spectrum. From the loop equation in time domain

$$\delta\omega_{\mathrm{LLO}}(t) = \delta\omega_{\mathrm{LO}}(t) - K_e e(t) \quad (28)$$

and assuming that $f_c \ll f_M$, which is usually the case in synchronous detection, we can extract the LLO frequency spectrum in closed loop operation. It is given, in the limit of high loop gain and introducing one-sided power spectral densities of fractional frequency, by

$$S_y^{\mathrm{LLO}}(f) \cong \frac{16\pi^2 f^2 T_I^2}{K^2 K_e^2 \omega_0^2 c_0^2 |H(0)|^2} S_y^{\mathrm{LO}}(f)$$
$$+ 2\sum_{k=1}^{+\infty} \frac{|c_{2k}|^2}{c_0^2} \frac{|H(2kf_M)|^2}{|H(0)|^2} S_y^{\mathrm{LO}}(2kf_M)$$
$$f < f_c. \quad (29)$$

We are interested here only in the aliased part of the spectrum. Introducing the expression of the transfer function $H(f)$, we obtain finally the following expression for the power spectral density of the LLO fractional frequency fluctuations due to aliasing:

$$S_{y,al.}^{\mathrm{LLO}}(f) \cong 2\sum_{k=1}^{+\infty} \frac{|c_{2k}|^2}{c_0^2} \mathrm{sinc}^2\left( 2k\pi \frac{T}{T_M} \right) S_y^{\mathrm{LO}}(2kf_M)$$
$$f < f_c \quad (30)$$

where $\mathrm{sinc}(x) = \sin(x)/x$ and $S_y^{\mathrm{LO}}(f)$ is the power spectral density of the LO fractional frequency fluctuations. This formula is the basic relationship to analyze the effect of any type of modulation-demodulation scheme on the LLO frequency stability.

## V. DISCUSSION

Without developing further (30), we can already point out some general features of the result.

### A. Modulation Frequency

Inspecting the term involving the ratio $T/T_M$ in (30), it follows in particular that for the condition $T = T_M/2$, this term will be equal to zero for any $k$ and as a result the power spectral density $S_{y,al.}^{\mathrm{LLO}}(f)$ will vanish. This property can be expressed differently in saying that if the modulation frequency is equal to the resonator linewidth, no frequency instability due to aliasing will be added to the LLO frequency. Another remarkable feature is that this property is true for *all usual modulation-demodulation schemes*, since the latter intervenes in the formula only through the Fourier coefficients $c_{2k}$.

### B. Interpretation of the Result

We can interpret this property as follows. In our model, the operation of demodulation which takes place in the synchronous detector is equivalent to multiplying the frequency fluctuations at the output of the resonator by a waveform that contains only even harmonics of the modulation frequency. As a result, all fluctuations whose Fourier frequencies are an even multiple of the modulation frequency will be downconverted into the bandpass of the frequency loop, giving rise to a spurious error signal introducing instabilities in the LLO frequency. The effect of the Ramsey resonator is to filter out, by averaging over the transit time, the LO fluctuations whose period is a submultiple of the transit time as can be seen from (9). Then, if the half-period of modulation corresponds to the transit time of the Ramsey resonator, all LO fluctuations whose Fourier frequencies are an
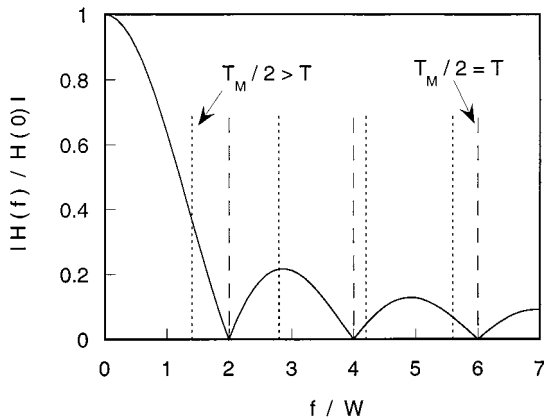
Fig. 2. Module of the normalized transfer function $H(f)/H(0)$ plotted versus the Fourier frequency $f$ normalized to the resonator linewidth $W$. This curve illustrates the filtering effect of the Ramsey resonator. The vertical lines show the location of the downconverted harmonics (... for $T_M/2 > T$, - - for $T_M/2 = T$).

even multiple of the modulation frequency will be cancelled at the output of the resonator as shown in Fig. 2.

It follows from what was said above that no downconversion into the bandpass is possible in that case or, in other words, that no frequency instability can be added to the LLO frequency by aliasing.

### C. Link with Previous Work

This property has previously been derived by Makdissi [7] in a different manner, namely in the context of an extension of the sensitivity function [2] from the pulsed to the continuously operated resonator. The generalization proposed in [7] is interesting since it starts from the well-known formalism developed by Dick [2], [3] and extends it to the continuous case. This formalism uses the two following notions of: 1) "phase steps" to describe an elementary phase fluctuation of the LO and 2) "sensitivity function" $g(t)$, describing the time-dependent response of the resonator to such "phase steps." The ideal situation for a frequency standard is to have a constant $g(t)$, that produces no aliasing. In a Ramsey resonator, $g(t)$ assumes nonzero values over the duration $T$ of the atomic transit, which is, in the case of a pulsed fountain, shorter than the interrogation cycle ($T_M/2$). With this formalism, the so-called "Dick effect" (aliasing produced by dead-times) may be seen as the consequence of the fact that if a "phase-step" occurs at some times (during a dead time), no error signal is produced ($g = 0$). Therefore, it is intuitively understandable that in a continuous fountain [9], [10] or a quasi-continuous fountain [13], [14] where atoms are always present in the resonator, no Dick effect will occur, while pulsed standards are subject to it. The "intermodulation effect" [5] has a similar origin but is slightly different, since it occurs even if there are no dead-times [15]. In the Dick formalism, this effect may be described mathematically by saying that $g(t)$ is not constant and physically by saying that the magnitude of the response of the resonator depends on "when" the phase-step occurs. Therefore, one may think that if the instantaneous signal is modulated [$I(t)$ or $s(t)$] either because of the modulation scheme (with sine-wave, for example, see [8]) or because the atomic beam is modulated [13], [14], the intermodulation effect

will appear and limit the stability, even in continuous fountains. As demonstrated in [7] and in this communication, this is not necessarily the case, since the signal [$I(t)$, $s(t)$ and $e(t)$] in a Ramsey resonator produced by any LO "phase-step", lasts $T$, the time spent by the atoms in the Ramsey resonator. If this time is equal to the interrogation cycle ($T_M/2$), the error signal $e(t)$ applied to the LO to correct for a phase step becomes independent of its time of occurrence, with any modulation form. The difficulty of applying the formalism developed by Dick to a continuous fountain lies in the fact that in this case the half-modulation period becomes comparable to $T$ and the error signal produced by a "phase-step" covers at least two interrogation cycles. This difficulty is overcome by the approach presented in this communication. Our approach is entirely based on a continuous operation of the resonator, considering the pulsed operation as a particular modulation of the continuous error signal (see Section VI below).

### D. The Case of Square-Wave Phase Modulation

We can see that for the condition $T = T_M/2$, square-wave phase modulation is expected to better cancel the LLO instability due to aliasing, since in this case the function $a(t)$ is constant and its Fourier coefficients $c_{2k}$ [in (21)] are all equal to zero ($k \neq 0$). This fact will probably be a useful feature in actual cases with atomic velocity distribution and finite Rabi interactions where the above condition can no longer be verified for all atoms.

### E. Limitations of the Model

In our simple model of a continuous fountain, we have made several idealizations in order to get a relatively simple analytical formula for the power spectral density of LLO frequency fluctuations due to aliasing phenomena. These idealizations set a limitation to the conclusions that we can draw from the formula (30). To better describe a real frequency standard and to get quantitative predictions, an extension of the model is necessary to include the main following points.

*1) Velocity Distribution of Atoms:* In our model, the atomic beam is monokinetic. In the real continuous fountain of cold cesium atoms, the initial longitudinal temperature of the beam is low [10], leading to a narrow velocity distribution (a few centimeters per second) around the mean launching velocity ($\sim$4 m/s). This spread of velocity induces a distribution of transit time in the Ramsey cavity. Even though small, this effect must be taken into account because in that case the condition $T = T_M/2$ is no longer verified for all atoms. The consequence is a residual degradation of the frequency stability, i.e., $S_{y,al.}^{LLO}(f) > 0$, that we have to estimate.

*2) Finite Rabi Interactions:* The finite interaction time of real frequency standards has not been taken into account in this study. This effect also has to be investigated, especially in the case of square-wave phase modulation where it is expected to have significant consequences. Indeed, in that case a small fraction of atoms can undergo a phase commutation in one or two interaction regions giving rise to sharp transients in the output signal of the resonator. The effect of these transients on the LLO frequency stability and the possibility to suppress them by blanking, which should reintroduce a small amount of aliasing

with a velocity distribution, are under study. It is worth noting that for the condition $T = T_M/2$ and with a monokinetic beam, such a blanking should not introduce any aliasing since in that case the symmetry properties of the function $a(t)$ are conserved.

*3) Phase Shift between Modulation and Demodulation:* In the present analysis, we have assumed that the demodulation waveform of the synchronous detector is in phase with the modulated signal. In the case of monokinetic beam, this is the optimum condition. But with a velocity distribution of atoms, the transit time $T$ and the detection delay $T_d$ between resonator and detector (not taken into account here) are not the same for all atoms, resulting in a modulated signal at the output of the resonator which is deformed with respect to that with monokinetic atoms. Therefore, the "optimum" phase shift between the modulated signal and the demodulation signal is not unambiguously defined. The effect of this phase adjustment on the stability is also worth being investigated.

*4) Bandwidth of the Detector:* The detector considered in the model is an ideal detector of infinitely wide bandwidth. The cutoff frequency in the high of a real detector will introduce a deformation of the modulated signal from the resonator (as suggested to us by C. Audoin). As a consequence, in the square-wave phase modulation the Fourier coefficients $c_{2k}$ of the function $a(t)$ will not vanish for the condition $T = T_M/2$, as discussed in Section V-D, since $a(t)$ is no longer a product of two perfect square waveforms. Even though expected to lead to a small degradation of the LLO frequency stability, this effect has to be investigated.

## VI. REDUCTION TO THE PULSED CASE

It is interesting now to compare the model to the well-known case of the Dick effect appearing in the pulsed mode operation of a cesium fountain frequency standard.

The periodic availability of the error signal can be taken into account in our model by a periodic sampling operation on $s(t)$, with period $T_c$. The error signal then takes the following form:

$$e(t) = -\frac{K}{2T_I} \int_{-\infty}^{t} \delta\phi_{\text{LO}}(t', T)a(t')u(t') \, dt' \qquad (31)$$

where $u(t)$ is a Dirac comb given by

$$u(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT_c). \qquad (32)$$

Within these assumptions and doing again the spectral analysis of the so-defined error signal, we find for the power spectral density of the LLO fractional frequency fluctuations

$$S_{y,\,al.}^{\text{LLO}}(f) \cong 2 \sum_{k=1}^{+\infty} \text{sinc}^2\left(k\pi\frac{T}{T_c}\right) S_y^{\text{LO}}(kf_c) \qquad f < f_c. \qquad (33)$$

This formula corresponds exactly to the well-known "Dick formula" in the case of a pulsed fountain. Indeed, the sinc term involved in (33) is equal to the ratio $g_k^2/g_0^2$ of the Fourier coefficients of the Fourier series development of the sensitivity function $g(t)$, as defined in [16].

## VII. CONCLUSION

A simple model of continuously operated fountain frequency standard has been developed to investigate possible degradations of the LLO frequency stability. This model has shown that if the half-period of modulation $T_M/2$ is equal to the transit time $T$, no frequency instability due to an aliasing effect appears. This is true for all usual modulation-demodulation schemes. The case of a pulsed frequency standard can be recovered by sampling the continuous error signal at the cycle frequency, leading to the well-known "Dick formula."

The extension of the model to better describe a real fountain frequency standard and to get quantitative predictions, by taking into account such effects as a velocity distribution of atoms and the finite Rabi interaction time, is under investigation. We expect that these effects will introduce only a small degradation of the frequency stability, not preventing from reaching the "shot noise" stability limit with a conventional quartz crystal oscillator and state-of-the-art frequency chain.

This model may also be applied to multipulse operation [13] or juggling [14] of future fountain frequency standards and sets a solid basis for future experimental investigations.

## REFERENCES

[1] G. Kramer, "Noise in passive frequency standards," in *Proc. Conf. Precise Electromagnetic Measurements*, 1974, pp. 157–159.

[2] G. J. Dick, "Local oscillator induced instabilities in trapped ion frequency standards," in *Proc. 19th Precise Time and Time Interval (PTTI)*, Redendo Beach, CA, 1987, pp. 133–147.

[3] G. J. Dick, J. D. Prestage, C. A. Greenhall, and L. Maleki, "Local oscillator induced degradation of medium-term stability in passive atomic frequency standards," in *Proc. 22th Precise Time and Time Interval (PTTI)*, Vienna, VA, 1990, pp. 487–508.

[4] G. Santarelli, C. Audoin, A. Makdissi, P. Laurent, G. J. Dick, and A. Clairon, "Frequency stability degradation of an oscillator slaved to a periodically interrogated atomic resonator," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 45, pp. 887–894, July 1998.

[5] C. Audoin, V. Candelier, and N. Dimarcq, "A limit to the frequency stability of passive frequency standards due to an intermodulation effect," *IEEE Trans. Instrum. Meas.*, vol. 40, pp. 121–125, Apr. 1991.

[6] A. Makdissi, E. de Clercq, C. Audoin, and A. Clairon, "Generalization of the Dick effect to continuous beam frequency standards," in *Proc. 1997 IEEE Int. Freq. Contr. Symp.*, 1998, pp. 263–269.

[7] A. Makdissi, "Traitement de Signal Appliqué aux Etalons Primaires de Fréquence: Amélioration de leur Exactitude et de leur Stabilité," Ph.D. dissertation, Univ. Paris-sud, Orsay, France, 1999.

[8] L. Lo Presti, D. Rovera, and A. De Marchi, "A simple analysis of the Dick effect in terms of phase noise spectral densities," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 45, pp. 899–905, July 1998.

[9] G. Dudle, G. Mileti, A. Joyet, E. Fretel, P. Berthoud, and P. Thoman, "An alternative cold cesium frequency standard: The continous fountain," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 47, pp. 438–442, Mar. 2000.

[10] G. Dudle, G. Mileti, A. Joyet, P. Berthoud, and P. Thomann, "Development of the ON/OFMET continous Cs fountain frequency standard: A progress report," in *Proc. 14th EFTF*, Torino, Italy, 2000.

[11] J. Vanier and C. Audoin, *The Quantum Physics of Atomic Frequency Standards*. New York: Adam Hilger, 1989, ch. 5.

[12] W. B. Davenport Jr. and W. L. Root, *An Introduction to the Theory of Random Signals and Noise*. Piscataway, NJ: IEEE Press, 1987, ch. 4.

[13] S. Ohshima, T. Kurosu, T. Ikegami, and Y. Nakadan, "Multipulse operation of cesium atomic fountain," in *Proceedings of the 5th Symposium on Frequency Standards and Metrology*. Singapore: World Scientific, 1995, pp. 60–65.

[14] C. Fertig and K. Gibble, "Laser-cooled $^{87}$Rb clock," *IEEE Trans. Instrum. Meas.*, vol. 48, pp. 520–523, Apr. 1999.

[15] J. Q. Deng, G. Mileti, R. E. Drullinger, D. A. Jennings, and F. L. Walls, "Noise considerations for locking to the center of a Lorentzian line," *Phys. Rev. A, Gen. Phys.*, vol. 59, no. 1, pp. 773–777, Jan. 1999.

[16] G. Santarelli, "Contribution à la Réalisation d'une Fontaine Atomique," Ph.D. dissertation, Univ. Paris VI, France, 1996.

**Alain Joyet** received the degree in physics from the University of Lausanne, Switzerland, in 1992. He spent one year as Assistant at the same university, then, after various activities, he joined the laser cooling group of ON in 1996 where he is currently pursuing the Ph.D. degree in physics in the field of cesium fountain primary frequency standard.

**Gregor Dudle** was born in Steinen, Switzerland, in 1965. He received the degree in physics from the Swiss Federal Institute of Technology (EPFL), Lausanne, Switzerland, in 1991. He worked at the Observatoire de Neuchâtel on laser cooling and received the Ph.D. from the University of Neuchâtel in 1996. Through a fellowship from the Royal Society, he spent one year at the National Physical Laboratory, Teddington, U.K., with the primary frequency standard group.

Since 1997, he has been with the Time and Frequency Laboratory of the Swiss Federal Office of Metrology, Wabern.

**Gaetano Mileti** was born in St-Aubin, Switzerland, in 1968. He received the engineering degree in physics from the Swiss Federal Institute of Technology (EPFL), Lausanne, in 1990, and the Ph.D. degree in physics from the Neuchâtel University, Switzerland, in 1995.

He has been with the Observatoire de Neuchâtel, Switzerland, since 1991. From 1995 to 1997, he was a Guest Scientist at NIST, Boulder, CO. His research interests are Rb (atomic vapor cell) and Cs (cold atomic beam) frequency standards, atomic laser cooling, laser spectroscopy, and high spectral resolution atmospheric Lidar techniques.

**Pierre Thomann** (M'90) received the degree in physics engineering and the Ph.D. degree in physics in 1978, both from the Swiss Federal Institute of Technology (EPFL), Lausanne, Switzerland.

He worked as a Postdoctoral Research Assistant one year at Oregon State University, Corvallis, and one year at JILA, University of Colorado, Boulder. From 1980 to 1991, he led several industrial developments in the field of atomic clocks (Rb, H-maser, Cesium). He then joined the Observatoire de Neuchâtel, Switzerland, as Head of the Cesium group, where his current interests are applications of laser-cooling of atoms, mostly in the field of time and frequency metrology.